Simple Current Extensions and Mapping Class Group Representations

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Abstract The conjecture of Fuchs, Schellekens and Schweigert on the relation of mapping class group representations and fixed point resolution in simple current extensions is investigated, and a cohomological interpretation of the untwisted stabilizer is given.

1 Introduction

In a recent paper [1], Fuchs, Schellekens and Schweigert presented an Ansatz to describe the modular properties of a CFT obtained by simple current extensions (for a review see e.g. [2]). The crucial step in this program is the understanding of fixed point resolution, for which they had to introduce the notion of *untwisted stabilizer*. Their Ansatz contains a set of matrices S^J associated to each simple current J of the extension, which should satisfy a number of nontrivial consistency conditions in order for the Ansatz to make sense. They have conjectured that these consistency conditions may be satisfied by choosing the quantities S^J to be the matrices that describe the transformation properties of the genus one holomorphic one-point blocks of the simple current J under the transformation S that exchanges the standard homology cycles of the torus.

The aim of this paper is to investigate the above questions in a general setting. We will show that starting from the S-matrix for one-point blocks one may define quantities that have very similar properties to those postulated by Fuchs, Schellekens and Schweigert . Moreover, we will get extra relations between these quantities exploiting their relationship to mapping class group representations and the extended fusion rule algebra [3]. Our results provide convincing evidence for the original conjecture that the quantities appearing in the FSS Ansatz are indeed related to the modular transformations of the one-point holomorphic blocks.

2 The module of one-point blocks and the generalized fusion algebra

For each primary field p let $\mathcal{V}(p)$ denote the space of genus one holomorphic one-point blocks of p. This space admits a finer decomposition

$$\mathcal{V}(p) = \bigoplus_q \mathcal{V}_q(p), \tag{1}$$

where the subspace $\mathcal{V}_q(p)$ consists of the blocks with the primary field q in the intermediate channel, and consequently its dimension is

$$\dim \mathcal{V}_q(p) = N_{pq}^q. \tag{2}$$

We introduce the notation $\mathcal{P}_q(p)$ for the operator projecting $\mathcal{V}(p)$ onto $\mathcal{V}_q(p)$.

It was shown in [3] that each $\mathcal{V}(p)$ affords a representation of the fusion algebra, i.e. there exist operators $\mathcal{N}_q(p) : \mathcal{V}(p) \to \mathcal{V}(p)$ satisfying

$$\mathcal{N}_q(p)\mathcal{N}_r(p) = \sum_s N_{qr}^s \mathcal{N}_s(p). \tag{3}$$

Moreover, it may be shown that

$$\mathrm{Tr}\left(\mathcal{N}_q(p)\right) = \sum_r N_{pr}^r \frac{S_{qr}}{S_{0r}}, \tag{4}$$

and that the operator $S(p) : \mathcal{V}(p) \to \mathcal{V}(p)$ implementing the modular transformation S on the space of one-point blocks diagonalizes simultaneously these generalized fusion rule matrices, i.e.

$$S^{-1}(p)\mathcal{N}_q(p)S(p) = \sum_r \frac{S_{qr}}{S_{0r}}\mathcal{P}_r(p). \tag{5}$$

Eq. (5) is the generalization of Verlinde's theorem to the space of one-point blocks. If α is a simple current, i.e. $S_{0\alpha} = S_{00}$, then $\mathcal{N}_\alpha(p)$ is invertible, and one has

$$\mathcal{N}_\alpha(p)^{-1}\mathcal{P}_q(p)\mathcal{N}_\alpha(p) = \mathcal{P}_{\alpha q}(p), \tag{6}$$

so $\mathcal{N}_\alpha(p)$ maps $\mathcal{V}_q(p)$ onto $\mathcal{V}_{\alpha q}(p)$.

The operators $S(p)$ and

$$T(p) = \kappa \sum_q \omega_q \mathcal{P}_q(p), \tag{7}$$

where $\omega_q = \exp(2\pi\imath\Delta_q)$ denotes the exponentiated conformal weight - or statistics phase - of the field q, and $\kappa = \exp(-\imath\pi c/12)$ is the exponentiated central charge of the theory, form a representation of the mapping class group $M_{1,1}$ of the one-holed torus, i.e. they satisfy the relation

$$S(p)T(p)S(p) = T^{-1}(p)S(p)T^{-1}(p). \tag{8}$$

Moreover,

$$S^4(p) = \omega_p^{-1}\mathbf{1}, \tag{9}$$

which shows that this is a projective representation of $SL(2,\mathbf{Z})$.

If $X \in M_{1,1}$ is any mapping class, then we shall denote by $\mathrm{Tr}_p (X)$ the trace of the operator representing it on $\mathcal{V}(p)$, and we define the quantity χ_p as [3]

$$\chi_p(X) = S_{0p} \sum_q \bar{S}_{pq} \mathrm{Tr}_q(X). \tag{10}$$

It is clear from its definition that χ_p is a class function on $M_{1,1}$, moreover it is normalized, i.e.

$$\chi_p(\mathbf{1}) = 1. \tag{11}$$

We shall need the following properties of χ_p in the sequel.

1. Invariance under simple currents : if α is a simple current, then

$$\chi_{\alpha p} (X) = \chi_p (X). \tag{12}$$

2. Cyclicity :

$$\chi_p \left(\mathcal{P}_q S \mathcal{P}_r S^{-1} \right) = \chi_q \left(\mathcal{P}_r S \mathcal{P}_p S^{-1} \right). \tag{13}$$

We shall also use the explicit expression of $\chi_p (X)$ for some mapping classes $X \in M_{1,1}$, as given in [3].

3   The FSS Ansatz

In [1] Fuchs, Schellekens and Schweigert have introduced, for any group $\mathcal{G}$ of integral spin simple currents, a set of unitary matrices S^α for each $\alpha \in \mathcal{G}$, whose index set consists of the fixed points of the simple current α. These matrices have to satify the following postulates :

1.

$$(S^\alpha T^\alpha)^3 = (S^\alpha)^2 \tag{14}$$

where T^α is a diagonal matrix with entries $T^\alpha_{pp} = \kappa \omega_p$ for the fixed points p of α.

2.

$$(S^\alpha)^2_{pq} = \eta^\alpha_p \delta_{p,\bar{q}} \tag{15}$$

for some phases η^α_p.

3.
$$\eta_p^\alpha \eta_p^\beta = \eta_p^{\alpha\beta} \tag{16}$$

for $\alpha, \beta \in U_p$, where the *untwisted stabilizer* U_p is to be defined below.

4.
$$\eta_{\bar{p}}^\alpha = \bar{\eta}_p^\alpha \tag{17}$$

5.
$$S_{p,q}^\alpha = S_{q,p}^{\alpha^{-1}} \tag{18}$$

6.
$$S_{\beta p,q}^\alpha = F(p, \beta, \alpha)\theta(q, \beta)S_{p,q}^\alpha, \tag{19}$$

for some (unknown) phases F, where

$$\theta(q, \beta) = \frac{\omega_{\beta q}}{\omega_\beta \omega_q} = exp\left(-2\pi \imath Q_\beta(q)\right) \tag{20}$$

is the exponentiated monodromy charge of the primary field q with respect to the simple current β.

7.
$$F(p, \alpha, \beta_1)F(p, \alpha, \beta_2) = F(p, \alpha, \beta_1\beta_2) \tag{21}$$

8.
$$F(p, \beta, \alpha) = F(p, \alpha, \beta^{-1}) \tag{22}$$

The last two conditions on the phases F show that the subset

$$U_p = \{\alpha \in \mathcal{S}_p \,|\, F(p, \alpha, \beta) = 1 \quad \forall \beta \in \mathcal{S}_p\}, \tag{23}$$

where $\mathcal{S}_p := \{\alpha \,|\, \alpha p = p\}$ is the stabilizer of p, is actually a subgroup of $\mathcal{S}_p$ whose index is a perfect square. Fuchs, Schellekens and Schweigert coined the term *untwisted stabilizer* for U_p, and showed that it is U_p, rather than $\mathcal{S}_p$, that governs the fixed point resolution. They proceeded on to show, that given a set of matrices S^α satisfying the above conditions, one can construct a new S-matrix for the resolved theory that meets the usual requirements - i.e. it is symmetric, unitary, generates an $SL(2, \mathbf{Z})$ representation with an appropriate T-matrix, its square is a permutation matrix -, with the possible exception of

giving integer fusion rule coefficients through Verlinde's formula. Clearly, the question is whether there exists some natural solutions to the above conditions, and if so whether they lead to a consistent resolved CFT, in particular do they yield integral fusion rule coefficients.

Eqs. (14) and (15) suggest that S^α and T^α might have some relation to mapping class group representations. This idea is supported by the observation that in the decomposition (1) of the space $\mathcal{V}(\alpha)$, only the fixed points of the simple current α appear, as a result of

$$N^q_{\alpha,q} = \delta_{q,\alpha q} \tag{24}$$

Moreover, the spaces $\mathcal{V}_q(\alpha)$ are all one dimensional, i.e. the space $\mathcal{V}(\alpha)$ indeed admits a basis labeled by the fixed points of α. In such a basis the matrix $T(\alpha)$ representing the transformation $T \in M_{1,1}$ would equal the matrix T^α of condition 1. above, and this suggests that in that basis the matrix of $S(\alpha)$ would equal S^α, because they satisfy similar conditions. It is this argument that led Fuchs, Schellekens and Schweigert to conjecture the equality of these two quantities, i.e. that

$$S^\alpha_{pq} = S(\alpha)_{pq} \tag{25}$$

The problem is, that to make Eq.(25) meaningful, one has to specify a preferred basis in the spaces $\mathcal{V}_q(\alpha)$ in order to define the matrix elements $S(\alpha)_{pq}$. At first sight it is not obvious how one should do that, but it turns at that there is indeed a canonical basis choice, up to some trivial indeterminacy that does not show up in the FSS Ansatz. The next section is devoted to a discussion of this canonical basis choice and a proof of the FSS conjecture that the resulting matrices $S(\alpha)_{pq}$ statisfy the requirements 1-8 above. In order to achieve this we shall introduce quantities related to the mapping class group action, and show that they have the same properties as their counterparts in the FSS Ansatz.

4 Simple current one-point blocks : the canonical basis choice and the mapping class group action

To begin with, let's recall the decomposition

$$\mathcal{V}(\alpha) = \bigoplus_p \mathcal{V}_p(\alpha) \tag{26}$$

of the space of genus 1 holomorphic one-point blocks of the simple current α, where $\dim \mathcal{V}_p(\alpha) = 1$ if $\alpha \in \mathcal{S}_p = \{\alpha \,|\, \alpha p = p\}$, and is 0 otherwise. We choose for each $\alpha \in \mathcal{S}_p$ a vector $e_p(\alpha) \in \mathcal{V}_p(\alpha)$.

Let's consider the quantity

$$\phi_p(\alpha, \beta) := \mathrm{Tr}_\alpha(\mathcal{N}_\beta \mathcal{P}_p). \tag{27}$$

From Eq. (6) we know, that $\mathcal{N}_\beta(\alpha)$ maps $\mathcal{V}_p(\alpha)$ onto $\mathcal{V}_{\beta p}(\alpha)$. This implies that $\phi_p(\alpha, \beta) = 0$ if p is not fixed by both α and β, while for $\alpha, \beta \in \mathcal{S}_p$ this means that

$$\mathcal{N}_\beta(\alpha) e_p(\alpha) = \phi_p(\alpha, \beta) e_p(\alpha), \tag{28}$$

and combining this with Eq.(3) we get at once that

$$\phi_p(\alpha, \beta\gamma) = \phi_p(\alpha, \beta)\phi_p(\alpha, \gamma). \tag{29}$$

Some other properties of $\phi_p(\alpha, \beta)$ follow from Eqs. (12) and (13) if we rewrite Eq. (27) as

$$\phi_p(\alpha, \beta) = \sum_{q,r} \theta(q, \alpha)\theta(r, \beta)\chi_p(\mathcal{P}_q S \mathcal{P}_r S^{-1}), \tag{30}$$

For example, we have for arbitrary simple currents α, β, γ

$$\phi_p(\beta, \alpha) = \phi_p(\alpha, \beta^{-1}) = \bar{\phi}_p(\alpha, \beta) \tag{31}$$

$$\phi_{\gamma p}(\alpha, \beta) = \phi_p(\alpha, \beta) \tag{32}$$

and the curious relation

$$\sum_p \theta(p, \alpha)\phi_p(\beta, \gamma) = \sum_p \theta(p, \beta)\phi_p(\gamma, \alpha) \tag{33}$$

Eq.(29) allows us to conclude that the subset

$$\mathcal{U}_p := \{\alpha \in \mathcal{S}_p \,|\, \phi_p(\alpha, \beta) = 1 \qquad \forall \beta \in \mathcal{S}_p\} \tag{34}$$

is actually a subgroup of $\mathcal{S}_p$, while as a consequence of (31), the index $[\mathcal{S}_p : \mathcal{U}_p]$ should be a perfect square. Note that

$$|\mathcal{U}_p| = \frac{1}{|\mathcal{S}_p|} \sum_{\alpha,\beta} \phi_p(\alpha, \beta). \tag{35}$$

Form Eq.(5) we know that

$$\mathcal{N}_\beta(\alpha)S(\alpha) = \sum_q \theta(q, \beta)S(\alpha)\mathcal{P}_q(\alpha), \tag{36}$$

and taking matrix elements of both sides in the basis $e_p(\alpha)$ leads to

$$\phi_p(\alpha, \beta)S_{pq}(\alpha) = \theta(q, \beta)S_{pq}(\alpha), \tag{37}$$

for $\alpha, \beta \in \mathcal{S}_p$. For a given p this implies in particular that $S_{pq}(\alpha) = 0$ for all q-s having 0 monodromy charge with respect to $\mathcal{G}$ whenever there exists a $\beta \in \mathcal{S}_p$ with $\phi_p(\alpha, \beta) \neq 1$, i.e. if α does not belong to $\mathcal{U}_p$. In other words, only those $S(\alpha)$-s will have non-vanishing matrix elements between p and q-s in $I_0^{\mathcal{G}} := \{q \,|\, \theta(q, \beta) = 1 \quad \forall \beta \in \mathcal{G}\}$ for which $\alpha \in \mathcal{U}_p$. This observation explains the role of the subgroup $\mathcal{U}_p$, which is of course nothing but the untwisted stabilizer of Fuchs, Schellekens and Schweigert .

To fully understand the relevance of ϕ_p and $\mathcal{U}_p$, we have to take a look now at the canonical basis choice. For each p, the space $\bigoplus_{\alpha \in \mathcal{S}_p} \mathcal{V}(\alpha)$ admits a binary bilinear associative operation $\bowtie$, such that the image of $\mathcal{V}_p(\alpha) \otimes \mathcal{V}_p(\beta)$ lies in $\mathcal{V}_p(\alpha\beta)$, i.e.

$$e_p(\alpha) \bowtie e_p(\beta) = \vartheta_p(\alpha, \beta)e_p(\alpha\beta), \tag{38}$$

where $\vartheta_p(\alpha, \beta)$ is some 2-cocycle of the stabilizer $\mathcal{S}_p$. In general $\bowtie$ is not commutative, because we have

$$e_p(\beta) \bowtie e_p(\alpha) = \phi_p(\alpha, \beta)e_p(\alpha) \bowtie e_p(\beta), \tag{39}$$

which implies at once that

$$\vartheta_p(\beta, \alpha) = \phi_p(\alpha, \beta)\vartheta_p(\alpha, \beta), \tag{40}$$

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i.e. ϕ_p is the so-called commutator cocycle of ϑ_p.

It is a well-known result of group cohomology, that the cocycle ϑ_p is trivial if and only if $\phi_p(\alpha, \beta) = 1$ for all α, β. But this condition holds for $\mathcal{U}_p$ by definition, consequently there exists some function ζ_p on $\mathcal{U}_p$ such that

$$\vartheta_p(\alpha, \beta) = \frac{\zeta_p(\alpha\beta)}{\zeta_p(\alpha)\zeta_p(\beta)} \tag{41}$$

for all $\alpha, \beta \in \mathcal{U}_p$. But then the rescaled basis vectors

$$\hat{e}_p(\alpha) = \zeta_p(\alpha)e_p(\alpha) \tag{42}$$

will satisfy

$$\hat{e}_p(\alpha) \bowtie \hat{e}_p(\beta) = \hat{e}_p(\alpha\beta) \tag{43}$$

for $\alpha, \beta \in \mathcal{U}_p$.

It is this last condition, Eq.(43) that fixes our canonical basis choice, i.e. the basis consisting of the vectors $\hat{e}_p(\alpha)$. Of course, there is still some indeterminacy left, because we are still free to rescale the vectors by

$$\hat{e}_p(\alpha) \mapsto \psi(\alpha)\hat{e}_p(\alpha), \tag{44}$$

where $\psi(\alpha)$ is some 1-dimensional character of the group $\mathcal{U}_p$, so that the resulting basis will still satisfy Eq.(43), but it is obvious that this indeterminacy does not show up in the FSS Ansatz.

Let's now take a look at

$$\eta(p, \alpha) := \mathrm{Tr}_\alpha(S^2\mathcal{P}_p) = \sum_q \theta(q, \alpha)\chi_q(S^2\mathcal{P}_p) \tag{45}$$

which obviously vanishes if p is not self-conjugate or not fixed by α, while for a self-conjugate fixed point its value is $\eta(p, \alpha) = S^2(\alpha)_{pp}$. Moreover, it follows from Eq. (45) that $\eta(p, \alpha) = 0$ if α has non-trivial monodromy with respect to any simple current, and that

$$\eta(\beta p, \alpha) = \eta(\bar{p}, \alpha) = \eta(p, \alpha). \tag{46}$$

The relation of $S(\alpha)^2$ to $\bowtie$ gives at once that

$$\eta(p, \alpha\beta) = \eta(p, \alpha)\eta(p, \beta) \qquad \text{for} \quad \alpha, \beta \in \mathcal{U}_p. \tag{47}$$

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From the results of [3] we can actually compute explicitly $\eta(p, \alpha)$, leading to

$$\eta(p, \alpha) = \nu_p \sum_{q,r} \theta(q, \alpha) N_{qr}^p S_{0q} S_{0r} \frac{\omega_q^2}{\omega_r^2}, \tag{48}$$

where ν_p is the Frobenius-Schur indicator [6] of the field p, which is 0 if p is not self-conjugate, and is ± 1 according to whether p is real or pseudo-real.

The next quantity we introduce is

$$\mu(p, \alpha) := \kappa^3 \omega_p \mathrm{Tr}_\alpha(S\mathcal{P}_p) = \kappa^3 \omega_p S_{pp}(\alpha). \tag{49}$$

First, we rewrite Eq. (49) in the form

$$\mu(p, \alpha) = \omega_p^{-1} \sum_{q,r} \omega_q^{-1} \theta(r, \alpha) \chi_r(\mathcal{P}_p S \mathcal{P}_q S^{-1}), \tag{50}$$

which in conjunction with Eq. (12) leads to

$$\mu(\beta p, \alpha) = \omega_\beta^{-1} \theta(p, \beta)^{-1} \mu(p, \alpha), \tag{51}$$

and

$$\mu(\bar{p}, \alpha) = \mu(p, \bar{\alpha}) = \mu(p, \alpha). \tag{52}$$

If we introduce the quantities

$$\mathcal{M}_k(\alpha, \beta) := \sum_p \omega_p^{1-k} \theta(p, \beta) \mu(p, \alpha), \tag{53}$$

then a simple argument involving Eq. (51) shows that

$$\mathcal{M}_k(\alpha, \beta \gamma^k) = \theta(\beta, \gamma) \omega_\gamma^k \mathcal{M}_k(\alpha, \beta), \tag{54}$$

and

$$\mathcal{M}_k(\alpha, \alpha \beta) = \omega_\alpha \mathcal{M}_k(\alpha, \beta). \tag{55}$$

The cyclicity property Eq. (13), together with Eqs. (30), (45) and (50), gives the sum rules

$$\mathcal{M}_0(\alpha, \beta) = \sum_p \omega_p^{-1} \phi_p(\alpha, \beta), \tag{56}$$

$$\mathcal{M}_4(\alpha, 0) = \kappa^6 \sum_p \omega_p \eta(p, \alpha), \tag{57}$$

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which connect $\mu(p, \alpha)$ with $\phi_p(\alpha, \beta)$ and $\eta(p, \alpha)$, while Eq. (54) and the explicit trace formulae of [3] give

$$\mathcal{M}_1(\alpha, \beta) \;=\; \kappa^9 \omega_\beta \sum_{p,q,r} N_{pq}^r S_{0p} S_{0q} S_{0r} \theta(p, \alpha) \frac{\omega_p^6 \omega_r^3}{\omega_q^2}, \tag{58}$$

$$\mathcal{M}_2(\alpha, \beta^2) \;=\; \kappa^9 \omega_\beta^2 \sum_{p,q,r} N_{pq}^r S_{0p} S_{0q} S_{0r} \theta(p, \alpha) \frac{\omega_p^4 \omega_r^4}{\omega_q^2}. \tag{59}$$

5   Conclusions

In the previous section we have investigated the structure of the space of genus one holomorphic one-point blocks of the simple currents, and have defined various quantities - such as ϕ and η - that characterize the action of the mapping class group on these spaces. We have seen that these quantities obey a host of non-trivial relations. We have also been able to specify a canonical basis choice - up to some trivial indeterminacy - in the space of one-point blocks, allowing us to give an invariant meaning to the matrix elements of the operators representing the mapping classes. Finally, we have introduced in an invariant way the subgroup $\mathcal{U}_p$ of the stabilizer $\mathcal{S}_p$, and gave a cohomological interpretation of its origin, related to the canonical basis choice.

Comparing our results with the postulates $1, \ldots, 8$ of Section 3, the truth of the FSS conjecture is obvious, since upon identifying S^α with $S(\alpha)$ - with respect to the canonical basis! -, $\phi_p(\alpha, \beta)$ is identified with $F(p, \alpha, \beta)$, $\mathcal{U}_p$ with U_p, and $\eta(p, \alpha)$ with η_p^α, and all of the conditions on these quantities are indeed fulfilled.

Unfortunately, the implementation of the canonical basis choice, as well as the explicit computation of the matrices $S(\alpha)$, is not a straightforward matter in general. To our best knowledge it had only been done in two class of models up to now, namely WZNW models through the use of orbit Lie-algebras [7], and holomorphic orbifold models through the application of the theory developed in [8]. It had been verified in these two cases by explicit numerical checks, that the FSS Ansatz does not only give a correct $SL(2, \mathbf{Z})$ representation, but indeed the one of the resolved theory.

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